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Detecting Singularities of Stewart Platforms

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Abstract. A Stewart platform, also known as a hexapod positioner, is a parallel manipulator using an octahedral assembly of struts. There are six independently actuated legs, whose lengths determine the position and orientation of the platform. These devices may display instabilities associated with architectural singularities and the purpose of the present study is to propose an approach for their detection. The main point is the formulation of the direct problem (given the leg lengths, find the position and orientation, velocity and acceleration of the platform) in an appropriate coordinate system based on quaternions.

Keywords. Stewart platform, parallel manipulator, quaternions, direct and inverse problems, architectural singularities

1 Introduction

The purpose of this study is to present methods allowing for the detection of singularities in a Stewart platform. These are points where the platform becomes uncontrollable, that is, for which its position will not be determined uniquely by fixing the lengths of the legs. To have an idea of what may happen, consider the simple situation where both the top and bottom platforms are two identical regular hexagons placed one on top of the other. Then the system has an extra degree of freedom and whatever the lengths of the six legs the platform will slide and collapse. In such a situation, we say that the architecture is singular – see [4], for instance. This is a well-known problem for systems of this type, and one of our aims is to provide efficient methods and tools to test the safety of a given platform.

In order to avoid this type of singularity, it is usual to consider a modified pair of hexagons such as that shown in Figure 1, where now the top platform is a rescaled and rotated copy of the

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bottom platform. Throughout this report we shall make this assumption, but other configurations may be studied using similar methods.

Although in this more general configuration the existence of singularities will not be as obvious as in the case exemplified above with two identical hexagons, they may still exist and part of the problems arising when designing a Stewart platform and its controller will be to ensure that these points lie outside the working area. If not, in spite of the fact that finding such a trajectory might be highly unlikely or even impossible, its existence will still affect the behaviour of the platform. Above all, it will not be possible to rule out completely a collapse due to sliding along such a path.

Before proceeding, let us make the following further assumptions:

A1. The control of the platform is done via the control of the lengths of the six legs.

A2. The joints are universal joints.

A3. The order of magnitude of the errors in the determination of the lengths of the six legs and in the joints may be considered to be negligible.

Having in mind the above assumptions, which basically allow us to rule out mechanical problems caused by insufficient precision in the components involved, we formulate the following working hypothesis which will dominate most of our study:

(H) A Stewart platform may only become uncontrollable if there exists a continuum of positions of the top platform corresponding to the same (fixed) values of the leg lengths.

We begin by considering the inverse and direct problems (Sections 2 and 3, respectively) and develop an approach based on an alternative formulation for the latter using quaternions (Section 3.2). In the last section we present some conclusions and suggestions which we believe to be important when working with this type of platforms. Finally we include a short appendix with a derivation of the dynamical equations in terms of quaternions.

2 The Inverse Problem: Lengths as a Function of Position

The Stewart platform considered has six degrees of freedom – see for instance, [2], p.279. We will first use the (standard) variables $x, y, z, pitch, roll$ and yaw, where x, y and z are the coordinates of the centre of the top platform, and $pitch, roll$ and yaw denote the Euler angles defining the inclination of this platform with respect to the bottom platform. As we will see, in order to study certain singular configurations, these variables are not always the best choice, and we will use a different approach in Section 3.2.

We take for the origin of our referential the centre of the circle that passes by all 6 points of the bottom platform. Assuming the radius of this circle to be one, the coordinates of the six points where the legs are supported are thus

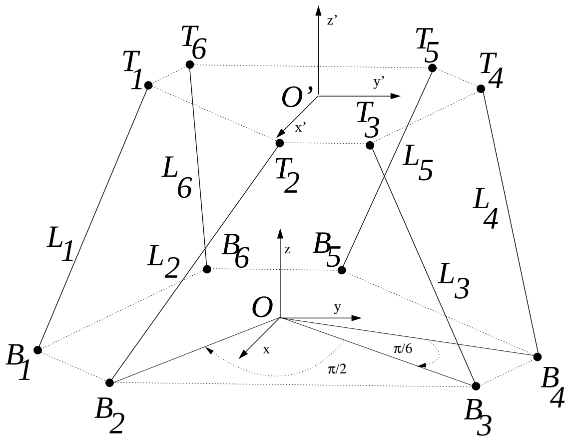

Figure 1: *The Stewart platform.*

$$B_1 = (\cos\frac{\pi}{12}, \sin\frac{\pi}{12}, 0), \qquad B_2 = (\cos\frac{7\pi}{12}, \sin\frac{7\pi}{12}, 0), \qquad B_3 = (\cos\frac{9\pi}{12}, \sin\frac{9\pi}{12}, 0),$$
$$B_4 = (\cos\frac{15\pi}{12}, \sin\frac{15\pi}{12}, 0), \qquad B_5 = (\cos\frac{17\pi}{12}, \sin\frac{17\pi}{12}, 0), \qquad B_6 = (\cos\frac{23\pi}{12}, \sin\frac{23\pi}{12}, 0).$$

Since the bottom and top platforms are related by a yaw rotation of π and a $2/3$ rescaling factor, we can use the matrix R defined by

$$\begin{pmatrix} -\frac{2}{3}\cos\psi\cos\theta & \frac{2}{3}[\cos\phi\sin\psi - \cos\psi\sin\theta\sin\phi] & -\frac{2}{3}[\sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta] & x \\ -\frac{2}{3}\cos\theta\sin\psi & -\frac{2}{3}[\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi] & \frac{2}{3}[\cos\psi\sin\phi - \cos\phi\sin\psi\sin\theta] & y \\ -\frac{2}{3}\sin\theta & \frac{2}{3}\cos\theta\sin\phi & \frac{2}{3}\cos\theta\cos\phi & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to find the six points of the top platform. The variables x, y, z represent the centre of the top platform and ψ, θ, ϕ the three Euler angles roll, pitch and yaw. Since a translation of the centre of the platform is not a linear application, in order to still be able to represent this transformation by a matrix we must consider a 4×4 matrix (a point (x_0, y_0, z_0) of $\mathbb{R}^3$ will be represented by $(x_0, y_0, z_0, 1)$).

To compute the leg lengths, we just have to compute the norms of the vectors $\mathbf{L}_1 = R(B_4) - B_1$, $\mathbf{L}_2 = R(B_5) - B_2$, $\mathbf{L}_3 = R(B_6) - B_3$, $\mathbf{L}_4 = R(B_1) - B_4$, $\mathbf{L}_5 = R(B_2) - B_5$, $\mathbf{L}_6 = R(B_3) - B_6$. Letting L_i, $i = 1, \cdots, 6$ be those norms, we have the following explicit formulae:

$$L_1^2(x, y, z, \psi, \theta, \phi) = \left(z + \tfrac{\sqrt{2}}{3}[\sin\theta - \cos\theta\sin\phi]\right)^2$$
$$+ \left(-\tfrac{1+\sqrt{3}}{2\sqrt{2}} + x + \tfrac{\sqrt{2}}{3}[\cos\psi\cos\theta - \cos\phi\sin\psi + \cos\psi\sin\theta\sin\phi]\right)^2$$
$$+ \left(\tfrac{1-\sqrt{3}}{2\sqrt{2}} + y + \tfrac{\sqrt{2}}{3}[\cos\theta\sin\psi + \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi]\right)^2,$$

$$L_2^2(x,y,z,\psi,\theta,\phi) = \left(z + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\sin\theta - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{-1+\sqrt{3}}{2\sqrt{2}} + x + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\cos\psi\cos\theta - \tfrac{1+\sqrt{3}}{3\sqrt{2}}[\cos\phi\sin\psi - \cos\psi\sin\theta\sin\phi]\right)^2$$
$$+\left(-\tfrac{1+\sqrt{3}}{2\sqrt{2}} + y + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\psi + \tfrac{1+\sqrt{3}}{3\sqrt{2}}[\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi]\right)^2,$$

$$L_3^2(x,y,z,\psi,\theta,\phi) = \left(z - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\sin\theta - \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{1}{\sqrt{2}} + x - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\cos\psi\cos\theta + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}[-\cos\phi\sin\psi + \cos\psi\sin\theta\sin\phi]\right)^2$$
$$+\left(-\tfrac{1}{\sqrt{2}} + y - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\psi + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}[\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi]\right)^2,$$

$$L_4^2(x,y,z,\psi,\theta,\phi) = \left(z - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\sin\theta + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{1}{\sqrt{2}} + x - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\cos\psi\cos\theta - \tfrac{-1+\sqrt{3}}{3\sqrt{2}}[-\cos\phi\sin\psi + \cos\psi\sin\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{1}{\sqrt{2}} + y - \tfrac{1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\psi + \tfrac{1-\sqrt{3}}{3\sqrt{2}}[\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi]\right)^2,$$

$$L_5^2(x,y,z,\psi,\theta,\phi) = \left(z + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\sin\theta + \tfrac{1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{-1+\sqrt{3}}{2\sqrt{2}} + x + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\cos\psi\cos\theta - \tfrac{1+\sqrt{3}}{3\sqrt{2}}[-\cos\phi\sin\psi + \cos\psi\sin\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{1+\sqrt{3}}{2\sqrt{2}} + y + \tfrac{-1+\sqrt{3}}{3\sqrt{2}}\cos\theta\sin\psi - \tfrac{1+\sqrt{3}}{3\sqrt{2}}[\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi]\right)^2,$$

$$L_6^2(x,y,z,\psi,\theta,\phi) = \left(z + \tfrac{\sqrt{2}}{3}[\sin\theta + \cos\theta\sin\phi]\right)^2$$
$$+\left(-\tfrac{1+\sqrt{3}}{2\sqrt{2}} + x + \tfrac{\sqrt{2}}{3}[\cos\psi\cos\theta + \cos\phi\sin\psi - \cos\psi\sin\theta\sin\phi]\right)^2$$
$$+\left(\tfrac{-1+\sqrt{3}}{2\sqrt{2}} + y + \tfrac{\sqrt{2}}{3}[\cos\theta\sin\psi - \cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi]\right)^2.$$

3 The Direct Problem: Position as a Function of the Six Lengths

This is a much more complex problem, as it involves inverting the expressions above in order to obtain x, y, z, ψ, θ and ϕ as functions of the leg lengths $L_i, i = 1, ..., 6$. Not only will this be much more demanding from a computational point of view, but in general we cannot expect these functions to be determined uniquely. More precisely, for a given set of leg lengths we will, in general, have more than one possible configuration of the platform.

3.1 The Jacobian

Let us consider the vector function $L : \mathbb{R}^6 \to \mathbb{R}^6$ with $L(x, y, z, \psi, \theta, \phi) \equiv (L_1, L_2, L_3, L_4, L_5, L_6)$. By computing the zeros of the Jacobian determinant of L, $J(L)$, we can find the points where L is not necessarily locally invertible, that is, the points of $\mathbb{R}^6$ where variations of $L_1, L_2, L_3, L_4, L_5, L_6$ can lead to more than one position of the top platform. Using the software *Mathematica* we were able to obtain an expression for J. However, this was too complex to be used analytically. In the case of some specific configurations it is still possible to determine several zeros of J and therefore possible problematic points.

3.2 Alternative Formulation Based on Quaternions

The formalism used in Sections 2 and 3.1 for the description of the relation between the leg lengths and the position of the platform makes it difficult to draw conclusions about the existence of the uncontrollable behaviour which has been observed. In order to study its existence we shall use unit quaternions to parameterise spatial rotations in three dimensions instead. In the following analysis we follow [5, 6].

Spatial rotations in three dimensions can be parameterised using both Euler angles (ϕ, θ, ψ) and unit quaternions $\mathbf{q} = (q_0, q_1, q_2, q_3)$, $||\mathbf{q}|| = 1$. A unit quaternion may be described as a vector $\mathbf{q}$ in $\mathbb{R}^4$ such that

$$\begin{aligned}\mathbf{q} &= (q_0, q_2, q_2, q_3),\\ \mathbf{q}^T\mathbf{q} &= q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1.\end{aligned}$$

The Euler angles are related to the unit quaternions by

$$\begin{aligned}\phi &= \arctan\left(\frac{2(q_0q_1 + q_2q_3)}{1 - 2(q_1^2 + q_2^2)}\right),\\ \theta &= \arcsin\left(2(q_0q_2 - q_3q_1)\right),\\ \psi &= \arctan\left(\frac{2(q_0q_3 + q_1q_2)}{1 - 2(q_2^2 + q_3^2)}\right),\end{aligned}$$

while the rotation matrix is given by

$$R = \begin{pmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & 2q_0^2 - 1 + 2q_3^2 \end{pmatrix}.$$

Consider the Stewart platform shown in Figure 1, where the two coordinate systems O and O' are fixed to the base and the mobile platforms. The platform geometry can be described by vectors $\mathbf{L}_i$, $i = 1, 2, \dots, 6$, defined by $\mathbf{L}_i = \mathbf{T}_i - \mathbf{B}_i$, $i = 1, 2, \dots, 6$, where $\mathbf{B}_i$ and $\mathbf{T}_i$ are the base and top vertex coordinates, respectively. We assume that these points are related by

$$\mathbf{T}_i = \mu A\mathbf{B}_i, \qquad i = 1, 2, \dots, 6,$$

where A is a 3×3 orthogonal matrix and $\mu \in (0, 1)$ is the rescaling factor. The coordinates of the base vertices are given by

$$\mathbf{B}_i = (x_i, y_i, 0), \qquad i = 1, 2, \dots, 6.$$

Given the position $\mathbf{P} = (x, y, z)$ and the transformation matrix R between the two coordinate systems, the leg vectors may be written as

$$\begin{aligned}\mathbf{L}_i &= R\mathbf{T}_i - \mathbf{B}_i + \mathbf{P}\\ &= (\mu RA - I)\mathbf{B}_i + \mathbf{P}, \qquad i = 1, 2, \dots, 6.\end{aligned}$$

So the length for each i-leg is given by

$$\mathbf{L}_i{}^T\mathbf{L}_i = ((\mu RA - I)\mathbf{B}_i + \mathbf{P})^T\left((\mu RA - I)\mathbf{B}_i + \mathbf{P}\right) \tag{1}$$

Given $\mathbf{q}$, A and $\mathbf{P}$ the leg lengths are given by

$$L_i = \sqrt{((\mu RA - I)\mathbf{B}_i + \mathbf{P})^T\left((\mu RA - I)\mathbf{B}_i + \mathbf{P}\right)}. \tag{2}$$

3.2.1 Closed Form Solutions

In the forward kinematics the six leg lengths L_i, $i = 1, 2, \ldots, 6$, are given, while R and $\mathbf{P}$ are unknown. Let $\mathbf{e}_x = (1,0,0)$, $\mathbf{e}_y = (0,1,0)$, $\mathbf{e}_z = (0,0,1)$ and expand (1) to get

$$\begin{aligned} L_i^2 &= \mathbf{P}^T\mathbf{P} + \mathbf{B}_i{}^T\left((\mu(RA)^T - I)(\mu RA - I)\right)\mathbf{B}_i \\ &+ 2\mathbf{B}_i{}^T(\mu(RA)^T - I)\mathbf{P}, \end{aligned} \tag{3}$$

or

$$\begin{aligned} L_i^2 &= \mathbf{P}^T\mathbf{P} + 2x_i\left(\mathbf{e}_x{}^T(\mu(RA)^T\mathbf{P} - \mathbf{P})\right) + 2y_i\left(\mathbf{e}_y{}^T(\mu(RA)^T\mathbf{P} - \mathbf{P})\right) \\ &- 2\mu\left[x_i^2(\mathbf{e}_x{}^T RA\mathbf{e}_x) + x_iy_i(\mathbf{e}_x{}^T RA\mathbf{e}_y + \mathbf{e}_y{}^T RA\mathbf{e}_x)\right. \\ &\left. + y_i^2(\mu\mathbf{e}_y{}^T RA\mathbf{e}_y)\right] + (1+\mu^2)(x_i^2 + y_i^2). \end{aligned} \tag{4}$$

Define now $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5, w_6)$ as

$$w_1 = \mathbf{P}^T\mathbf{P} \tag{5}$$

$$w_2 = 2\mu\mathbf{e}_x{}^T((RA)^T\mathbf{P} - \mathbf{P}) \tag{6}$$

$$w_3 = 2\mu\mathbf{e}_y{}^T((RA)^T\mathbf{P} - \mathbf{P}) \tag{7}$$

$$w_4 = -2\mu\mathbf{e}_x{}^T RA\mathbf{e}_x \tag{8}$$

$$w_5 = -2\mu\left(\mathbf{e}_x{}^T RA\mathbf{e}_y + \mathbf{e}_y{}^T RA\mathbf{e}_x\right) \tag{9}$$

$$w_6 = -2\mu\mathbf{e}_y{}^T RA\mathbf{e}_y. \tag{10}$$

and $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5, d_6)$.

$$d_i = L_i^2 - (1+\mu^2)(x_i^2 + y_i^2), \qquad i = 1, 2, \ldots, 6. \tag{11}$$

Then relation (4) can be written as a linear system with the form

$$Q\mathbf{w} = \mathbf{d}, \tag{12}$$

where the matrix Q is given by

$$Q = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & y_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & y_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & y_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4y_4 & y_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5y_5 & y_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & y_6^2 \end{pmatrix}.$$

Note that if the base vertices are in a circle $x_i^2 + y_i^2 = r$ for some r, then $\det Q = 0$.

In the next sections we will show that one can obtain the rotation matrix R and the position $\mathbf{P}$ in terms of the solution $\mathbf{w} = (w_1, w_2, \ldots, w_6)$ of the linear system given by (12). The solution to the forward kinematics problem naturally divides into two cases, namely, a non-singular case where $\det Q \neq 0$ and a singular case where $\det Q = 0$.

In the singular case, we obtain for a given set of length legs $L_1, \ldots, L_6$, a singular solution parameterised by a scalar parameter. These solutions describe a curve in the position and rotation spaces in which the platform moves without changing the values of the leg lengths.

3.2.2 Non-Singular Case

In the case where the six base vertices are not on a quadratic curve, one gets $\det Q \neq 0$, and so the solution of (12), $\mathbf{w} = (w_1, w_2, w_3, w_4, w_5, w_6)$, may be obtained from

$$\mathbf{w} = Q^{-1} d.$$

Equations (5), (6) and (7) determine the position $\mathbf{P} = (x, y, z)$, while equations (8), (9) and (10) give us the rotation parameters, namely, $\mathbf{q}$.

To determine the rotation parameters consider the equations

$$w_4 = -2\mu \left(2{q_1}^2 + 2{q_0}^2 - 1\right) \tag{13}$$

$$w_5 = -8\mu q_1 q_2 \tag{14}$$

$$w_6 = -2\mu \left(2{q_2}^2 + 2{q_0}^2 - 1\right), \tag{15}$$

which are obtained from (8), (9) and (10), respectively. Eliminating q_0, one gets

$$\begin{aligned} {q_1}^2 - {q_2}^2 &= -(w_4 - w_6)/(4\mu) \\ q_1 q_2 &= -w_5/(8\mu). \end{aligned}$$

Let

$$\alpha = \frac{w_4 - w_6}{4\mu}, \qquad \beta = -\frac{w_5}{8\mu}.$$

Then the above equations can be written as

$$\begin{aligned} q_1^4 + \alpha q_1^2 - \beta^2 &= 0 \\ q_2^4 - \alpha q_2^2 - \beta^2 &= 0. \end{aligned}$$

Thus

$$q_1^2 = \frac{-\alpha + \gamma}{2} \tag{16}$$

and

$$q_2^2 = \frac{\alpha + \gamma}{2}, \tag{17}$$

where

$$\gamma = \sqrt{\alpha^2 + 4\beta^2}. \tag{18}$$

Substituting q_1 and q_2 in the first equation in (13) and taking into account that $\mathbf{q}^T\mathbf{q} = 1$ yields

$$q_0^2 = \frac{1}{2} - \frac{w_4}{4\mu} + \frac{\alpha - \gamma}{2}, \tag{19}$$

$$q_3^2 = \frac{1}{2} + \frac{w_4}{4\mu} - \frac{\alpha + \gamma}{2}. \tag{20}$$

Provided that equations (16) to (20) have two solutions each, this would give a total of eight different quaternions.

To determine the position, consider the equations

$$\mathbf{u}^T = 2\mu\mathbf{e}_x{}^T((RA)^T - I),$$
$$\mathbf{v}^T = 2\mu\mathbf{e}_y{}^T((RA)^T - I).$$

Thus

$$\mathbf{P}^T\mathbf{P} = w_1, \tag{21}$$

$$\mathbf{u}^T\mathbf{P} = w_2, \tag{22}$$

$$\mathbf{v}^T\mathbf{P} = w_3. \tag{23}$$

Clearly (22) and (23) represent two planes whose intersection is the line given by

$$\mathbf{P} = \mathbf{r}_0 + t\mathbf{r}_1, \tag{24}$$

where t is a real parameter and the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ are given by

$$\mathbf{r}_0 = \frac{(\mathbf{v}^T\mathbf{v})w_2 - (\mathbf{u}^T\mathbf{v})w_3}{(\mathbf{u}^T\mathbf{u})(\mathbf{v}^T\mathbf{v}) - (\mathbf{u}^T\mathbf{v})^2}\mathbf{u} - \frac{-(\mathbf{u}^T\mathbf{v})w_2 + (\mathbf{u}^T\mathbf{u})w_3}{(\mathbf{u}^T\mathbf{u})(\mathbf{v}^T\mathbf{v}) - (\mathbf{u}^T\mathbf{v})^2}\mathbf{v},$$
$$\mathbf{r}_1 = \frac{\mathbf{u} \times \mathbf{v}}{||\mathbf{u} \times \mathbf{v}||}.$$

This line intersects the sphere given by equation (21) at two points

$$P_\pm = \mathbf{r}_0 \pm t^*\mathbf{r}_1,$$

where

$$t^* = \sqrt{w_1 - \mathbf{r}_0^T\mathbf{r}_0}.$$

Note that in order for $P_\pm$ to exist one must have

$$w_1 \geq \mathbf{r}_0^T\mathbf{r}_0. \tag{25}$$

In this way we have determined both R and $\mathbf{P}$, there being a total of eight possible different solutions for a given set of leg lengths.

3.2.3 Singular Case

In this case we assume that all points belong to a circle $x_i^2 + y_i^2 = 1$, $i = 1, 2, \ldots, 6$. Without loss of generality we shall take r to be one. In this case the matrix

$$Q = \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1y_1 & 1 - x_1^2 \\ 1 & x_2 & y_2 & x_2^2 & x_2y_2 & 1 - x_2^2 \\ 1 & x_3 & y_3 & x_3^2 & x_3y_3 & 1 - x_3^2 \\ 1 & x_4 & y_4 & x_4^2 & x_4y_4 & 1 - x_4^2 \\ 1 & x_5 & y_5 & x_5^2 & x_5y_5 & 1 - x_5^2 \\ 1 & x_6 & y_6 & x_6^2 & x_6y_6 & 1 - x_6^2 \end{pmatrix} \tag{26}$$

is singular, that is, $\det Q = 0$ and in fact, except for highly degenerate cases, the rank of Q is five. This will be the case if $x_i^2 + y_i^2 = 1$, $i = 1, 2, \ldots, 6$, and $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$, $i, j = 1, 2, \ldots, 6$, corresponding to the Braikenridge-Maclaurin construction [1].

This fact enables us to explicitly compute the LU factorisation of the matrix Q in terms of the coordinates of the vertices of the base (x_i, y_i), $i = 1, 2, \ldots, 6$. The linear system $Q\mathbf{w} = d$ may thus be written in the form

$$U\mathbf{w} = L^{-1}d, \tag{27}$$

where $\det L = 1$ and U is a rank 5 matrix. The solution of (27) is given in terms of a solution $(w_1, w_2, w_3, w_4, w_5)$ depending on the value of w_6, which we take to be a free parameter. So the expressions given by (16), (17), (19) and (20) can be used to determine the values of the quaternion $\mathbf{q}$, the rotation matrix, and the point $\mathbf{P}$ as a function of the free parameter w_6. Note that existence of these solutions also depends on the inequality given by (25).

To illustrate the method, we shall now present an example in which we compute a solution of this type explicitly for a given set of leg lengths.

3.2.4 An Example

Consider the platform given in Figure 1 and assume $\mu = 2/3$. The base vertices coordinates are given by

$$B_i = (x_i, y_i, 0) = (\cos\theta_i, \sin\theta_i, 0), \qquad i = 1, 2, \ldots, 6 \tag{28}$$

where θ is given by

$$\theta = \Big(0.262, 1.8326, 2.3562, 3.927, 4.4506, 6.0214\Big). \tag{29}$$

Assume that

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Substituting (28) in (26) yields

$$Q = LU = \begin{pmatrix} 1.0 & 0.966 & 0.259 & 0.933 & 0.25 & 0.067 \\ 1.0 & -0.259 & 0.966 & 0.067 & -0.25 & 0.933 \\ 1.0 & -0.707 & 0.707 & 0.5 & -0.5 & 0.5 \\ 1.0 & -0.707 & -0.707 & 0.5 & 0.5 & 0.5 \\ 1.0 & -0.259 & -0.966 & 0.067 & 0.25 & 0.933 \\ 1.0 & 0.966 & -0.259 & 0.933 & -0.25 & 0.067 \end{pmatrix},$$

and one gets

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1.0 & 1 & 0 & 0 & 0 & 0 \\ 1.0 & 1.366 & 1 & 0 & 0 & 0 \\ 1.0 & 1.366 & 3.7321 & 1 & 0 & 0 \\ 1.0 & 1.0 & 3.7321 & 1.366 & 1 & 0 \\ 1.0 & 0.0 & 1.0 & 0.366 & 1.0 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1.0 & 0.966 & 0.259 & 0.933 & 0.25 & 0.067 \\ 0 & -1.2247 & 0.707 & -0.866 & -0.5 & 0.866 \\ 0 & 0 & -0.518 & 0.75 & -0.067 & -0.75 \\ 0 & 0 & 0 & -2.049 & 1.183 & 2.049 \\ 0 & 0 & 0 & 0 & -0.866 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$

with $(L_1, L_2, L_3, L_4, L_5, L_6) = (0.870, 0.820, 0.820, 0.840, 0.850, 0.889)$, one gets,

$$L^{-1}d = \begin{pmatrix} -0.688 \\ -0.0845 \\ 0.0474 \\ -0.113 \\ 0.0273 \\ 0 \end{pmatrix}.$$

The quaternions are obtained in terms of w_6 by

$$\begin{aligned} \bar{q}_0^2 &= 0.485 - 0.375w_6, \\ \bar{q}_1^2 &= 0.00124, \\ \bar{q}_2^2 &= 0.0288, \\ \bar{q}_3^2 &= 0.485 + 0.375w_6. \end{aligned}$$

This implies that $|w_6| \leq 1.2933$. In Figures 2 and 3 we show the values of the quaternion solution $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$ and of t^*, and the values of P_+ as a function of w_6, respectively. From Figure 3 we see that the solution where the values of the quaternion $\mathbf{q}$ and position $\mathbf{P}$ varies continuously with w_6 depends on t^*.

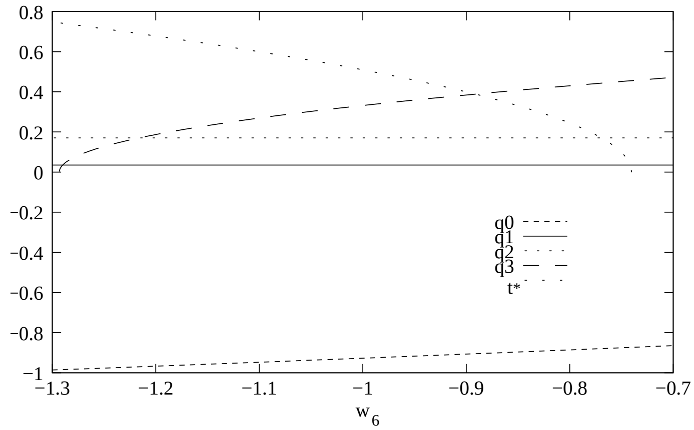

Figure 2: *The positive quaternion solution* $(\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3)$ *as a function of* w_6, *for the fixed set of leg lengths* $(L_1, L_2, L_3, L_4, L_5, L_6) = (0.870, 0.820, 0.820, 0.840, 0.850, 0.889)$.

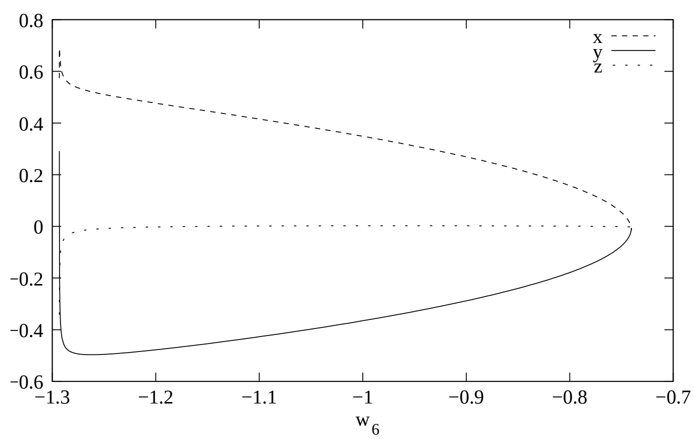

Figure 3: *The position solution* $P_+ = (x, y, z)$ *as a function of* w_6.

4 Conclusions and Recommendations

This study supports the hypothesis that the collapse of this type of platforms is caused by the existence of a continuum of solutions. However, the existence of such a class of solutions as described in Section 3 depends drastically on the geometry of the base and top platforms. Thus, given a particular configuration, a more thorough study needs to be performed and, in particular, the specific characteristics of the platform need to be taken into account as described below.

The reasons for this shape dependence are related to the fact that although the singularity of the determinant described in Section 3.2.3 depends only on the geometry of the base plate, the existence of one-parameter families of solutions with fixed leg lengths will depend on the specific geometry of the top plate.

If the top platform under consideration is not obtained by a simple rescaling and rotation of the bottom platform, further tests are needed to determine whether or not a continuum of solutions will exist in that case (this may be achieved by means of a projective transformation taking a circle into a conic section, thus, extending the results already obtained in this report). However, the fact that we were able to find these for a wide class of top platforms and with leg-lengths within the working area, leads us to believe that hypothesis (H) holds.

Furthermore, this would also explain why it might be difficult or even impossible to bring a platform back into the working area once it collapses, without dismantling it. This is an effect known to have happenned in practice and the main point is that it would be almost impossible to find such a trajectory by trial and error alone. Besides, the momentum that the system would gather once there is one extra degree of freedom might also be sufficient to force the platform past what might be called a *bridge*, that is, it might allow it to jump from one continuum to another – it is not clear either if or when this may happen, as it depends on several other considerations.

This hypothesis also explains another anomaly which is sometimes found during the testing of Stewart platforms, namely, the fact that a platform may be found to be in a different position from that which was predicted by the controlling software. Due to the dynamics of the platform, passing close to a singular point in a continuum might not be sufficient to allow the platform to collapse, as the speed will not, in general, be tangent to the continuum – to our knowledge, actual collapses have been observed mainly while platforms are at rest. This effect might, however, be sufficient to deflect the trajectory – more precisely, to affect the component of the velocity along the direction of the continuum – thus changing the final position of the platform. As a first step towards the study of the dynamics, see the Appendix where the derivation of the equations of motion for the platform is given in terms of quaternions.

A summary of our recommendations is the following:

1. One basic conclusion is that due to the complexity of the problem the formulation used is of fundamental importance; at this level, and besides using the matrix R in the formulation of the inverse problem, we strongly recommend the usage of the formulation based on quaternions which was presented on Section 3.2

2. For each specific platform, a study similar to that presented in Section 3.2.3 should be carried out to ascertain the existence of continua of solutions in that case. This mainly implies adapting the computations in that section to take the specific shape and dimensions into consideration.

3. Carry out extensive tests to ascertain whether there exist mismatches between the actual position of the platform and the position predicted by the software, even under reduced maximum leg-length. If our hypothesis are correct, this provides an indirect way of detecting whether there might still exist continua under the reduced working regime.

4. Implement a feedback mechanism for the position, in order to ensure that the software is in tune with the actual position of the platform. This would also help ensure that the software is well adapted to the platform.

At another level, it is clear that an important step in the future of any such project will be the design of trajectories with specific predefined characteristics. This will be a major effort and will require the introduction of new techniques.

References

[1] Coxeter, H.S.M.: *Projective Geometry*, 2nd Edition, Springer-Verlag, New York (1987). 74

[2] Craig, J.: *Introduction to Robotics: Mechanics and Control*, 3rd edition, Prentice Hall (2005). 67

[3] Dasgupta, B. and Mruthyunjaya, T.: Closed-form dynamic equations of the general Stewart Platform through the Newton-Euler approach, *Mechanism and Machine Theory* **33** no.7 (1998), 993-1012.

[4] Ma, O. and Angeles, J.: Architecture Singularities of Platform Manipulators, *Proceedings of the 1991 IEEE International Conference on Robotics and Automation*, 1542-1547. 66

[5] Ji, P. and Wu, H.: A Closed-Form Kinematics Solutions for the $6-6^p$ Stewart Platform, *IEEE Transactions on Robotics and Automation* **17** no.4 (2001), 522-526. 70

[6] Diebel, J.: Representing Attitude: Euler Angles, Quaternions, and Rotation Vectors 70

[7] Harib, K. and Srinivasan, K.: Kinematic and dynamics analysis of the Stewart platform-based machine toll structures, *Robotica* **21** (2003), 541-554. 79

Appendix: Dynamical Behaviour

In order to establish the equations determining the motion of the platform, (see for instance [7]) one should also consider the formulation in terms of quaternions. This will provide a strong mathematical basis for future developments.

Consider the expression for the leg lengths position (1) and lengths given by (2) and define the vector $\mathbf{n}_i = \mathbf{L}_i / L_i$, for $i = 1, 2, \ldots, 6$. The velocity of the point $\mathtt{L}_i$ is obtained by differentiating (1) with respect to time to obtain

$$\dot{\mathbf{L}}_i = \dot{\mathbf{P}} + \omega \times \tilde{R} \cdot \mathbf{B}_i, \quad i = 1, 2, \ldots, 6, \tag{30}$$

where ω is the angular velocity vector and $\tilde{R} = \mu R A$. Then

$$\dot{L}_i = \mathbf{L}_i \cdot \mathbf{n}_i = \dot{\mathbf{P}} \cdot \mathbf{n}_i + \omega \cdot \left(\tilde{R}\mathbf{B}_i\right) \times \mathbf{n}_i, \quad i = 1, 2, \ldots, 6. \tag{31}$$

In matrix form the system (31) takes the form

$$\dot{\mathbf{l}} = J^{-1} \begin{bmatrix} \dot{\mathbf{P}} \\ \omega \end{bmatrix}, \tag{32}$$

where J is the Jacobian matrix of the form

$$J^{-1} = \begin{bmatrix} \mathbf{n}_1^T & \left(\tilde{R}\mathbf{B}_1 \times \mathbf{n}_1\right)^T \\ \vdots & \vdots \\ \mathbf{n}_6^T & \left(\tilde{R}\mathbf{B}_6 \times \mathbf{n}_1\right)^T \end{bmatrix}. \tag{33}$$

The angular velocity ω is related to the quaternion $\mathbf{q}$ by the transformation

$$\begin{bmatrix} 0 \\ \omega \end{bmatrix} = 2\mathcal{Q}^T(\mathbf{q})\dot{\mathbf{q}}, \tag{34}$$

where

$$\mathcal{Q}(\mathbf{q}) = \begin{bmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & q_3 & -q_2 \\ q_2 & -q_3 & q_0 & q_1 \\ q_3 & q_2 & -q_1 & 1_0 \end{bmatrix}. \tag{35}$$

Plugging (34) into (32) one gets

$$\dot{\mathbf{L}} = J^{-1} J_q^{-1} \begin{bmatrix} \dot{\mathbf{P}} \\ \dot{\mathbf{q}} \end{bmatrix}, \tag{36}$$

where

$$J_q^{-1} = \begin{bmatrix} \mathbf{I}_{3\times 3} & \mathbf{0}_{3\times 4} \\ \mathbf{0}_{4\times 3} & 2\mathcal{Q}^T(\mathbf{q}) \end{bmatrix} \tag{37}$$

The platform is in a singular position whenever the determinant of $\tilde{J}^{-1} = J^{-1}J_q^{-1}$ is zero. These zeros occur when $\det J^{-1} = 0$ (configuration singularities) or $\det J_q^{-1} = 0$ (formulation singularities). Configuration singularities are difficult to find analytically.

The equations of motion may be obtained by differentiating equation (36) with respect to time yielding

$$\ddot{\mathbf{L}} = \tilde{J}^{-1}\begin{bmatrix}\ddot{\mathbf{P}}\\ \ddot{\mathbf{q}}\end{bmatrix} + \frac{d\tilde{J}^{-1}}{dt}\begin{bmatrix}\dot{\mathbf{P}}\\ \dot{\mathbf{q}}\end{bmatrix}. \tag{38}$$