Interactive Freeform Design of Tensegrity

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Abstract. *In this paper, we propose a novel interactive method for flexibly designing tensegrity structures under valid force equilibriums. Unlike previous form-finding techniques that aim to obtain a unique solution by fixing some parameters such as the force densities and lengths of elements, our method provides a design system that allows a user to continuously interact with the form within a multidimensional solution space. First, a valid initial form is generated by converting a given polygon mesh surface into a strut-and-cable network that follows the original surface, and the form is then perturbed to attain an equilibrium state through a two-step optimization of both node coordinates and force densities. Then, the form can be freely transformed by using a standard 2D input device while the system updates the form on the fly based on the orthogonal projection of a deformation mode into the solution space. The system provides a flexible platform for designing a tensegrity form for use as a static architectural structure or a kinetic deployable system.*

1 Introduction

1.1 Background

A *tensegrity* structure is a tension-stabilized, self-standing spatial structure composed of struts (compression members) and cables (tension members), where every strut is isolated or connected to the other struts only via cables. Different forms of tensegrity structures have been explored through experiments by sculptors and architects such as Kenneth Snelson and Buckminster Fuller. In addition to being a form of expression, the tensegrity has been applied to a lightweight structural system or a deployable system in the design of architecture. Since a tensegrity structure is in a self-equilibrium state because of stressed cables, its geometry is constrained intrinsically. Therefore, when tensegrity structures are to be used in architectural and spatial designs, a computational form-finding approach is useful.

One of the bases for solving pin-jointed wire-frame structures in equilibrium is the *force density method* introduced by [Linkwitz and Schek 1971], which successfully solves the form of a prestressed cable network by solving only one linear system. However, typically, the direct application of the force density method to tensegrity structures yields poor results; the configuration degenerates to a point, line, or plane. This is because the free-standing self-equilibrium state is produced only by a nontrivial singular set of force densities.

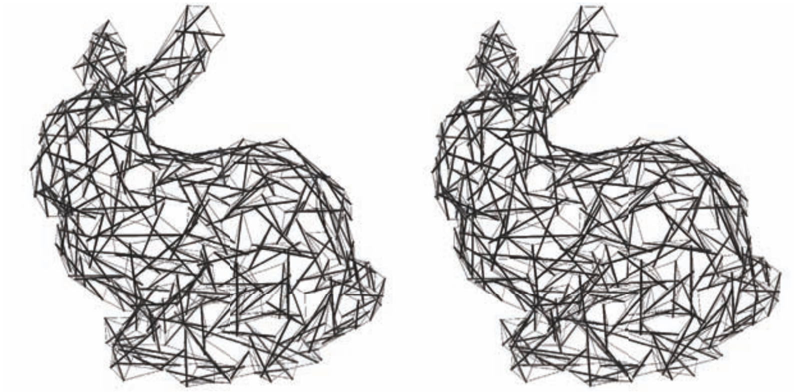


Figure 1: Tensegritized version of a simplified Stanford Bunny derived by using our algorithm (cross-eyed stereo).

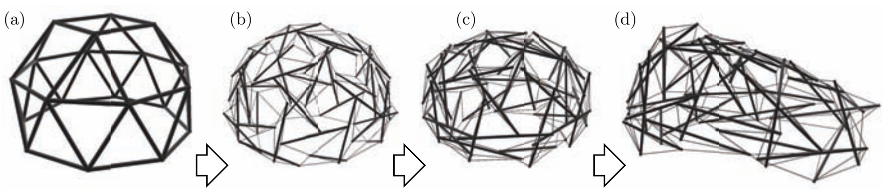


Figure 2: The process. (a) Given polygonal mesh. (b) Generated topology. (c) Valid initial configuration in equilibrium. (d) A freeform variation of (c).

Different numerical form-finding methods for tensegrity structures have been investigated by many researchers; for a more systematic review in this field, see [Tibert and Pellegrino 2003]. One of the conventional approaches for solving the singularity of tensegrity is to find a set of force density values that ensure a valid spatial configuration, for example, the approach in the *adaptive force density method* proposed by [Zhang and Ohsaki 2006]. Another approach is that used in the dynamic relaxation method for solving a nonlinear problem to converge to one of the solutions through an iterative physical or pseudo-physical simulation [Motro 1984; Miki and Kawaguchi 2010]. However, existing approaches use some predefined parameters or an objective function, e.g., constant lengths, force densities, or a pseudo-energy definition, in order to converge to some “unique” solution. However, from the design point of view, such predefined parameters are not directly relevant to the design, but more restrict the design freedom.

1.2 Overview

In this paper, we propose a novel computational method that first facilitates freeform designing of tensegrity structures through the construction of a single-layered tensegrity structure that follows the given surface and freeform manipulation based on user interaction (an example is shown in figure 1). Instead of obtaining a "unique" solution, we construct a responsive environment that allows users to interactively explore the continuous solution space of valid tensegrity structures. By separating the mandatory equilibrium constraints from the optional geometric constraints, the designers have more freedom in selecting different geometric constraints, such as equal-tension conditions. The procedure for the construction of a tensegrity structure in our method is as follows (Figure 2).

1. *Topology Generation Step*: The topology and the initial configuration are constructed from a given polygonal mesh. See Section 2 for details.
2. *Form-finding Step*: The initial configuration is modified to have a valid equilibrium state by performing two-step iterative optimization. See Section 3 for details.
3. *Interactive Design Step*: A user can interactively modify the form by directly dragging the structure displayed on the screen. The system ensures that the structure satisfies the equilibrium conditions by orthogonally projecting the deformation at each frame. See Section 4 for details.

2 Topology Generation

A topology of a strut-and-cable network can be constructed as the initial configuration for form-finding in multiple ways. Here, we describe a method that converts a general given polygonal mesh into a strut-and-cable network that forms a single layer of a surface. The process is as follows (Figure 3 top).

1. Isolate each edge and independently rotate it by θ $(-90^\circ < \theta < 90^\circ)$ about an axis passing through its midpoint parallel to the surface normal. θ is a parameter selected by the user. These edges are defined as struts.
2. Connect the endpoints with cables to form each m-gonal closed loop corresponding to an m-valency vertex.
3. Connect the endpoints with cables to form each n-gonal closed loop corresponding to an n-gonal facet.

The above construction process ensures that each node is shared by one strut and four cables. This converts a regular tetrahedron into a well-known six-strut icosahedral tensegrity structure (Figure 3 bottom).

3 Form-Finding

The initial configuration of the tensegrity structure is then modified to satisfy the geometry and force constraints of a tensegrity structure. The configuration of the

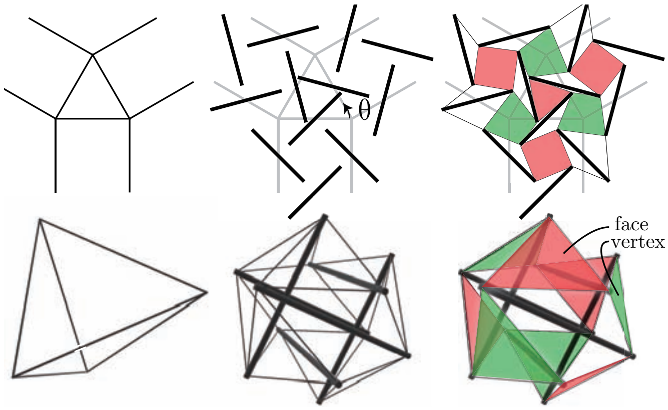


Figure 3: Top: Process of topology generation from a polygon mesh. Bottom: Conversion of regular tetrahedron into a regular icosahedral tensegrity structure in the case of $\theta = 45°$.

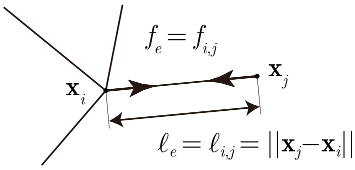


Figure 4: Notation of the parameters.

structure is represented by the coordinate $\mathbf{x}_i = \{x_i, y_i, z_i\}^\mathsf{T}$ of each node i and the force density $w_e = w_{i,j} = f_{i,j}/\ell_{i,j} = f_{i,j}/\left\|\mathbf{x}_i - \mathbf{x}_j\right\|$ of each segment e incident to vertices i, j, where $f_e = f_{i,j}$ is the axial force applied to the segment (Figure 4). Overall, the configuration is represented by a $3V$ vector $\mathbf{X} = \left(\mathbf{x}_1^\mathsf{T}, \cdots, \mathbf{x}_V^\mathsf{T}\right)^\mathsf{T}$ and E vector $\mathbf{W} = (w_1, \cdots, w_E)^\mathsf{T}$, where V is the number of vertices (nodes) and E is the number of edges (segments). In our constructed configuration, these numbers are related as follows: $V = 2S$ and $E = 5S$, where S is the number of struts, i.e., the number of edges on the original polygon mesh. The equilibrium at each node is then represented as

$$\mathbf{f}_i = \sum_{j \text{incident to } i} f_{i,j} \frac{(\mathbf{x}_j - \mathbf{x}_i)}{\left\|\mathbf{x}_j - \mathbf{x}_i\right\|} = \sum_{j \text{incident to } i} w_{i,j} (\mathbf{x}_j - \mathbf{x}_i) = \mathbf{0}. \tag{1}$$

In the force density method, $\mathbf{W}$ is a fixed constant, and the system is linear. In our method, $\mathbf{W}$ becomes a variable, and the system is nonlinear. Since the sign of $w_{i,j}$ describes whether the element is a tension or compression member, we assert inequalities

$$w_{i,j} \begin{cases} < -\alpha & \text{for a strut} \\ > \alpha & \text{for a wire} \end{cases} \tag{2}$$

Here, the positive value α can be set arbitrarily in the sense that an arbitrary scaling of $w_{i,j}$ is consistent with equation (1). We set $w_{i,j}$ proportional to a scale factor; the detailed reason is for this setting is discussed in Section 4.2. Our problem definition is as follows: solve $\mathbf{f}_i = \mathbf{0}$ for every node i with respect to $\mathbf{x}_i$ and $w_{i,j}$ while satisfying inequality (2).

The solution is not unique, and we find a solution close to the original configuration by the following two steps to make $U(\mathbf{X},\mathbf{W}) = \sum_i \|\mathbf{f}_i\|^2 + \sum_e \max(0, \alpha - |w_e|)^2 = 0$:

1. minimize U with respect to $\mathbf{W}$ for a fixed geometry $\mathbf{X}$.
2. minimize U with respect to $\mathbf{W}$ and $\mathbf{X}$ using an iterative method wherein the initial configuration is given by step 1.

In the first step, the number of variables is $E = 5S$ and $3V = 6S$ target equilibrium equations exist. Therefore, the minimization problem converges to the least-squares solution of an over-constrained system. In the second step, the number of variables is $E + 3V > 3V$, and U minimizes to 0; thus, equation (1) is satisfied at every node. The conjugate gradient method is used for solving these optimization problems. We have observed that this two-step approach avoids the degenerate trivial configurations, e.g., the one where all points are located on a point.

4 Interactive Freeform Editing

4.1 Projection Method

Our method enables a free exploration of the continuous solution space, instead of just obtaining a single solution. In order to find a valid deformation within the solution space, we use a projection-based method using the Moore-Penrose generalized inverse matrix [Penrose 1955] of a Jacobian matrix, which has been applied for shape analysis of transformable structures [Hangai and Kawaguchi 1991].

The configuration of the structure is represented by a $3V + E$ vector $\begin{bmatrix} \mathbf{X} \\ \mathbf{W} \end{bmatrix}$. Since each node satisfies equation (1), the variables are constrained by a $3V$-vector equation $\mathbf{F}(\mathbf{X},\mathbf{W}) = \left\{\mathbf{f}_0^\mathsf{T}, \cdots, \mathbf{f}_V^\mathsf{T}\right\}^\mathsf{T} = \mathbf{0}$. This forms an underdetermined system where a multidimensional continuous solution space exists.

A first-order deformation in the solution space can be represented by using the Jacobian matrix of the constraints:

$$[\mathbf{C}] \begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{W} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{F}}{\partial \mathbf{X}} & \frac{\partial \mathbf{F}}{\partial \mathbf{W}} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{W} \end{bmatrix} = \mathbf{0} \tag{3}$$

The elements of the Jacobian matrix are calculated as follows:

$$\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j} = \begin{cases} -\sum_{k \text{incident to } i} w_{i,k} & \text{if } i = j \\ w_{i,j} & \text{if } j \text{ is incident to } i \\ 0 & \text{else} \end{cases} \tag{4}$$

$$\frac{\partial \mathbf{f}_i}{\partial \mathbf{w}_{j,k}} = \begin{cases} \mathbf{x}_k - \mathbf{x}_i & \text{if } i = j \text{ (or } \mathbf{x}_j - \mathbf{x}_i \text{ if } i = k) \\ 0 & \text{else} \end{cases} \tag{5}$$

One of the solutions of Equation (3) is calculated using the Moore-Penrose generalized inverse $\mathbf{C}^+$ of the constraint matrix $\mathbf{C}$ as follows:

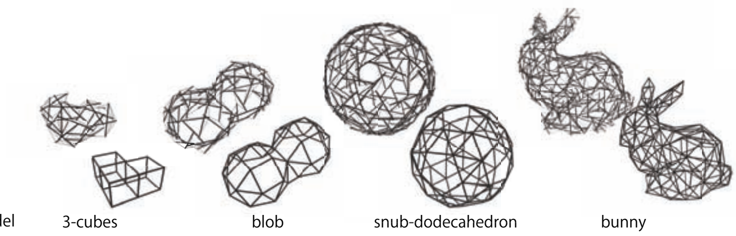
$$\begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{W} \end{bmatrix} = \left(\mathbf{I} - \mathbf{C}^+\mathbf{C}\right) \begin{bmatrix} \Delta\mathbf{X}_0 \\ \Delta\mathbf{W}_0 \end{bmatrix} \tag{6}$$

Here, $\begin{bmatrix} \Delta\mathbf{X}_0 \\ \Delta\mathbf{W}_0 \end{bmatrix}$ is an arbitrary vector. Since the multiplication of $\left(\mathbf{I} - \mathbf{C}^+\mathbf{C}\right)$ in equation (6) represents the orthogonal projection to the constrained subspace, the solution $\begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{W} \end{bmatrix}$ is the nearest valid defromation to a given $\begin{bmatrix} \Delta\mathbf{X}_0 \\ \Delta\mathbf{W}_0 \end{bmatrix}$.

We develop an interactive system based on this projection procedure. In the system, the user can freely select and drag the points on the rendered tensegrity structures through a 2D input device to apply a deformation, e.g., a point translation or a weighed point-set translation or rotation. Before the deformation is applied to the current state, the deformation vector $\begin{bmatrix} \Delta\mathbf{X}_0 \\ \mathbf{0} \end{bmatrix}$ is substituted with $\begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{W} \end{bmatrix}$ to get the constrained valid next state. Since the abovementioned Euler integration accumulates error, the residual is eliminated using Newton-Raphson's method for each step. The generalized inverse solution for each step is calculated using the conjugate gradient method.

4.2 Weight of Parameters

Since the orthogonal projection modifies the deformation to the least norm solution, it is essential that the configuration $\begin{bmatrix} \Delta\mathbf{X} \\ \Delta\mathbf{W} \end{bmatrix}$ have a consistent dimension. Therefore, we represent the parameters $\mathbf{W}$ in the length unit (i.e., we use $[\mathrm{m}]$ instead of $[\mathrm{N/m}]$). In order to attain scale invariance, we set the initial values of the parameters proportional to some scale factor such as the average length of the initial form. The self-equilibrium force can be uniformly scaled independent of the form; however, this results in a change in the response to a given input deformation. When the values of force density are scaled down, the proportion of force densities becomes easier to be changed, so the equilibrium state can adapt to a given deformation; as a relative result, the form follows the original user's input deformation more strongly. This scaling factor can also be controlled during freeform editing.



original model	3-cubes	blob	snub-dodecahedron	bunny
number of nodes	56	204	300	600
number of struts	28	102	150	300
number of cables	112	408	600	1200
initial form-finding	0.5 [s]	2.6 [s]	3.3 [s]	490 [s]
interaction frame rate	24 [fps]	7.5 [fps]	4.8 [fps]	0.85 [fps]

Figure 5: Amount of time required for the calculation.

5 Design System

5.1 Interface

A software system is implemented using C++ with OpenGL for graphics and BLAS (with Intel MKL) for linear algebra. The user can first import a mesh model in the OBJ format, let the system convert it to a valid tensegrity form, and then edit the form interactively by directly selecting and dragging the form. Figure 8 shows a screenshot of the system. The system runs in either the edit mode or the simulation mode, and the user toggles to switch between the modes. In the edit mode, interactive freeform editing described in Section 4 is performed, and the dragging of the form causes changes in the shape and force of the structure. In the simulation mode, the system calculates the deformation under the length constraints that fix the struts' lengths and prevent the cables from extending. We can check whether the tensegrity structure is stable by examining it in the simulation mode.

5.2 Computation Time

Figure 5 shows the amount of time required for performing calculations for four models with different complexities on a laptop PC with an Intel Core i5-2502M CPU (2.50 GHz dual core). For models with up to 150 struts, form-finding is sufficiently fast and the form modification is performed at an interactive frame rate. The slower convergence of the Stanford Bunny model with 300 struts in the initial form-finding step is due to an oscillation that occurred when solving the inequality conditions. However, the interactivity in the form modification process had an acceptable value of 0.85 fps, in which we can use the system repeatedly to obtain the desired configuration before actually constructing a model.

Figure 6: Image of a tensegrity structure that fits the ground.

6 Optional Geometric Constraints for Designs

In our method, we can add optional geometric constraints or penalty forces arbitrarily, provided that adequate degrees of freedom remain in the solution space. This is an advantage of our method over methods based on the convergence to a unique solution or a pseudo-physical simulation, in which cases adding constraints or penalty forces would destroy the equilibrium. Now, we consider several important geometric constraints and demonstrate their use for design and fabrication of a tensegrity structure.

Fixing to Reference: The system enables the fixing of selected points to reference points or to the ground plane. Figure 6 shows an example of a design in which several points are fixed on the ground so that the whole structure is placed on the ground, as intended. Here, the whole structure is merely placed on the ground; therefore, the joints at the ground bear the weight and the external load only (the prestresses are balanced within the structure).

Fixed Length: The lengths of elements can be fixed by adding a constraint $\ell_{i,j} = const$ for each rigidized edge i, j, whose derivative is given by $\partial\ell_{i,j}/\partial\mathbf{x}_i = (\mathbf{x}_i - \mathbf{x}_j)/\ell_{i,j}$. An interesting usage of the fixed length constraint is the design of a transformable tensegrity structure. After designing one configuration, we select some cables, define them as the actuators, and keep the lengths of the other struts and cables fixed. A transformation that can be obtained in the design system can be realized in the constructed structure by controlling the lengths of the actuators (Figure 7).

Equal Tension: By introducing the equal-tension constraint represented by $\ell_{i,j}w_{i,j} - \ell_{j,k}w_{j,k} = 0$, we can simplify the fabrication process. In the construction process of a tensegrity structure, we install prestresses by tensioning each cable.

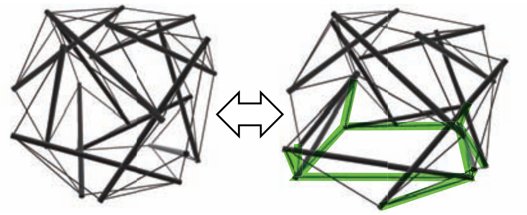


Figure 7: Transformable tensegrity. Two configurations can be transformed among themselves by controlling the lengths of actuator cables shown in green.

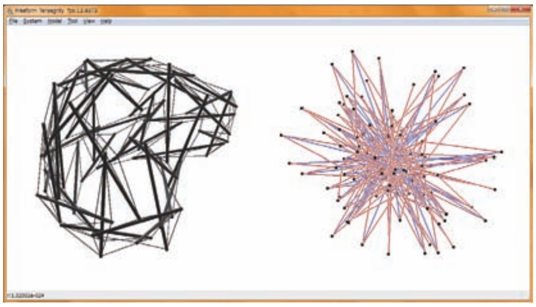


Figure 8: Screenshot of the system. A form with equal tension constraint applied to each cable loop (left) and its force diagram (reciprocal figure) (right).

However, adjustment of the tension becomes difficult for a complex model because the change in the tension of one of the cables affects the tensions of the others. Here, if we know in advance that adjacent cables are to have the same tension, then we can replace them with a continuous cable passing through a hole at the end of the strut; therefore, the joint structure is also simplified. This is because an equal-tension cable does not apply a sliding force to the joint.

An effective way of setting equal tension constraints is to group each closed loop that is originally derived from the same vertex or facet in the topology generation step (Section 2) and assign the same tension to each group. Although this constraint is a trade-off between the freedom of design and the simplicity of construction, we could still obtain an interesting asymmetric form by using the equal-tension constraint (Figure 8).

Intersection Avoidance of Struts: In the actual construction, the intersection of struts must be avoided. In our method, we can simply use a penalty force calculated from the distance between intersecting struts and use them as the deformation mode. This is in contrast to a form-finding based on pseudo-physical simulation, in which introducing a penalty force would affect the equilibrium.

7 Conclusion

We proposed a novel interactive method for freely designing tensegrity structures. First, we discussed the process of constructing a valid initial tensegrity structure from an arbitrary polygon mesh based on topology generation and form-finding through a two-step optimization. Then, we presented a method for interactive freeform editing based on solving the equilibrium using projection to the solution space. Our method succeeds in manipulating a tensegrity structure at an interactive frame rate for a model with 750 segments by using a standard laptop PC. We further realized optional geometric constraints, including the fixed length constraint, which enables the design of transformable tensegrity structures, and the equal-tension constraint, which simplifies the actual fabrication process. The proposed method ensured only the equilibrium and left the evaluation of stability to the user by providing the simulation mode in the system. In future, we intend to investigate the theoretical approach for ensuring the stability and stiffness of the structure.

Acknowledgments

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